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RESONANT MOTIONS OF A SPACECRAFT RELATIVE TO THE CENTER OF MASS SITUATED AT THE TRIANGULAR LIBRATION POINT OF THE SYSTEM EARTH—MOON*

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Resonant rotational motions of a rigid body situated at the triangular libration point of the restricted, circular three-body problem are investigated. The integrable Delaunay—Hill averaging scheme is used to study the long term periodic effects in the motion of a body relative to the intrinsic center of mass.

1. Equations of perturbed motion. We introduce the following Cartesian coordinate systems: inertial $GXYZ$ system; rotating $Gxyz$ system with the origin at the center of mass of the bodies M_0 (Earth) and M_1 (Moon) the axes Gx and Gy of which are situated in the orbital plane of these bodies, with the Gx-axis coinciding with the line OO_1 passing through the centers of mass O and O_1 of the bodies M_0 and M_1 and pointing towards the body M_1; $Sxyz$ system with the origin at the center of mass of the body M (spacecraft) and the axes parallel to the corresponding axes of the $Gxyz$ coordinate system; $S\xi\eta\zeta$ system the axes of which are directed along the principal central axes of inertia of the body M, and A, B, C are the principal central moments of inertia of the body M.

We describe the rotational motion of the satellite using the osculating Andoyer elements /1,2/

$$G, \theta, \rho, l, g, h \tag{1.1}$$

referred to the rotating $Gxyz$-axes. Here G is the value of the vector of kinetic moment of the rotational motion of the body, θ is the angle between the vector $\mathbf{G}$ and the $S\zeta$-axis of the body, ρ is the angle between $\mathbf{G}$ and the normal to the orbital plane, h is the longitude of the ascending node of the intermediate plane P normal to the vector $\mathbf{G}$, l is the angle of natural rotation of the body counted from the plane P, and g is the longitude of the ascending node of the $S\xi\eta$-plane of the body on the intermediate plane. Using the variables (1.1), we write the equations of rotational motion in the following form /1,3/:

$$\frac{dG}{dt} = \frac{\partial U}{\partial g}, \quad \frac{d\theta}{dt} = G\sin\theta\sin l\cos l\left(\frac{1}{A}-\frac{1}{B}\right)+\frac{1}{G}\operatorname{ctg}\theta\frac{\partial U}{\partial g}-\frac{1}{G}\operatorname{cosec}\theta\frac{\partial U}{\partial l} \tag{1.2}$$

$$\frac{d\rho}{dt} = \frac{1}{G}\operatorname{ctg}\rho\frac{\partial U}{\partial g}-\frac{1}{G}\operatorname{cosec}\rho\frac{\partial U}{\partial h}, \quad \frac{dh}{dt} = -n_0+\frac{1}{G}\operatorname{cosec}\rho\frac{\partial U}{\partial \rho}$$

$$\frac{dg}{dt} = G\left(\frac{\sin^2 l}{A}+\frac{\cos^2 l}{B}\right)-\frac{1}{G}\operatorname{ctg}\theta\frac{\partial U}{\partial \theta}-\frac{1}{G}\operatorname{ctg}\rho\frac{\partial U}{\partial \rho}, \quad \frac{dl}{dt} = G\cos\theta\left(\frac{1}{C}-\frac{\sin^2 l}{A}-\frac{\cos^2 l}{B}\right)+\frac{1}{G}\operatorname{cosec}\theta\frac{\partial U}{\partial \theta}$$

where U is the force function of the problem. Restricting ourselves to the second harmonic of the force function of the problem, we obtain the following trigonometric representation of the function U suitable for further investigation (the summation is carried out over k_1, k_2 and k_3):

$$U = \lambda\sum U_{k_1,k_2,k_3}(\theta,\rho,\delta)\{\cos[k_1 l + k_2 g + k_3(h-\Psi)] + \nu\cos[k_1 l + k_2 g + k_3(h+\Psi)]\} \tag{1.3}$$

$$\lambda = \frac{3}{16}n_0^2(A-B)\frac{1}{1+\nu}, \quad \delta = \frac{A-C}{A-B}, \quad \Psi = 60^\circ, \quad k_1 = 0,1,\ldots,\infty, \; k_2 = 0, \pm 2, \; k_3 = 0, \pm 1, \pm 2.$$

The coefficients U_{k_1,k_2,k_3} are known functions of the variables and of the constant dynamic parameter δ, n_0 denotes the mean orbital motion of the moon and ν is the ratio of the Moon and Earth masses.

2. General integral of the Delaunay—Hill averaged equations of motion. We use the Delaunay—Hill method to investigate the resonant modes of motion of a rigid satellite. Assuming that the inertia ellipsoid of the body is nearly spherical, we introduce a small parameter $\sigma = |A-B|/|B|$. Then the Eqs. (1.2) will assume a standard form (in the sense of application of the asymptotic methods) for which various averaging schemes (including the Delaunay—Hill averaging scheme) have been given a mathematical proof and a perturbation theory developed /4/.

In the present paper we consider the case of a resonant rotation of a body for which the condition $k_1'n_0 - k_2'n_1^{(0)} \sim \sigma$ of commensurability holds, where n_0 and $n_1^{(0)}$ denote the unperturbed velocities of the orbital and rotational motion and k_1' and k_2' are the commensurability

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indices. We introduce the Delaunay anomaly $D = k_1'g + k_2'h$ and construct, following the known rules, the equations of the intermediate rotational motion. We have

$$\frac{dG'}{dt} = k_1' \frac{\partial \langle U\rangle}{\partial D}, \quad \frac{d\theta'}{dt} = G' \sin\theta' \sin l' \cos l' \left(\frac{1}{A} - \frac{1}{B}\right) + \frac{k_1'}{G} \operatorname{ctg}\theta' \frac{\partial \langle U\rangle}{\partial D} - \frac{1}{G} \operatorname{cosec}\theta' \frac{\partial \langle U\rangle}{\partial \rho'} \tag{2.1}$$

$$\frac{d\rho'}{dt} = \frac{k_1'}{G} \operatorname{ctg}\rho' \frac{\partial \langle U\rangle}{\partial D} - \frac{k_2'}{G} \operatorname{cosec}\rho' \frac{\partial \langle U\rangle}{\partial D}, \quad \frac{dh'}{dt} = -n_0 + \frac{1}{G} \operatorname{cosec}\rho' \frac{\partial \langle U\rangle}{\partial \rho'}$$

$$\frac{dg'}{dt} = G' \left(\frac{\sin^2 l'}{A} + \frac{\cos^2 l'}{B}\right) - \frac{1}{G'} \operatorname{ctg}\theta' \frac{\partial \langle U\rangle}{\partial \theta'} - \frac{1}{G'} \operatorname{ctg}\rho' \frac{\partial \langle U\rangle}{\partial \rho'}$$

$$\frac{dl'}{dt} = G' \cos\theta' \left(\frac{1}{C} - \frac{\sin^2 l'}{A} - \frac{\cos^2 l'}{B}\right) + \frac{1}{G'} \operatorname{cosec}\theta' \frac{\partial \langle U\rangle}{\partial \theta'}$$

$$\frac{dD}{dt} = -k_2'n_0 + k_1'G' \left(\frac{\sin^2 l'}{A} + \frac{\cos^2 l'}{B}\right) - \frac{k_1'}{G'} \operatorname{ctg}\theta' \frac{\partial \langle U\rangle}{\partial \theta'} + \frac{1}{G'} \operatorname{cosec}\rho' (k_2' - k_1' \cos\rho') \frac{\partial \langle U\rangle}{\partial \rho'}$$

Here $G', \theta', \rho', l', g', h', D$ are the elements of the intermediate motion and $\langle U\rangle$ is the Delaunay—Hill averaged force function which has the form

$$\langle U\rangle = \sigma\lambda_0 \{(1 + \nu)(U_{000} + U_{200} \cos 2l') + U_{022} [\cos 2(D - \Psi) + \nu \cos 2(D + \Psi)] + U_{222} [\cos 2(l' + D - \Psi) + \nu \cos 2(l' + D + \Psi)] + U_{2-2-2} [\cos 2(l' - D + \Psi) + \nu \cos 2(l' - D - \Psi)]\} \tag{2.2}$$

$$U_{000} = -2(1 - 2\delta)[\sin^2\theta' + (1 - {}^3/_2 \sin^2\theta') \sin^2\rho'], \quad U_{200} = \sin^2\theta' (3\sin^2\rho' - 2), \quad U_{022} = {}^1/_2 \sin^2\theta' (1 + \cos\rho')^2 (1 - 2\delta)$$

$$U_{222} = -{}^1/_4 (1 + \cos\theta')(1 + \cos\rho')^2, \quad U_{2-2-2} = -{}^1/_4 (1 - \cos\theta')^2 (1 + \cos\rho')^2, \quad \lambda_0 = \frac{3}{16} \frac{n_0^2}{(1 + \nu)} B\varepsilon_0, \quad \varepsilon_0 = \begin{cases} 1, & A > B \\ -1, & A < B \end{cases}$$

In what follows, we shall omit for simplicity the primes accompanying the corresponding variables.

Equations (2.1) and (2.2) cannot be reduced directly to quadratures. This can however be done in a particular case, important in the study of synchronous satellites, by introducing additional simplifications. In the case of commensurability when $k_1' = k_2' = 1$, it can be shown that the equations (2.1) admit the solution $\theta = \pi/2,\ l = 0$ (an analog of the plane motion in the restricted three-body problem), and equations for the variables G, ρ, g, h, D for an independent system

$$\frac{dG}{dt} = \frac{\partial W}{\partial D}, \quad \frac{d\rho}{dt} = \frac{1}{G} \operatorname{cosec}\rho (\cos\rho - 1) \frac{\partial W}{\partial D}, \quad \frac{dh}{dt} = -n_0 + \frac{1}{G} \operatorname{cosec}\rho \frac{\partial W}{\partial \rho}, \tag{2.3}$$

$$\frac{dg}{dt} = \frac{G}{B} - \frac{1}{G} \operatorname{ctg}\rho \frac{\partial W}{\partial \rho}, \quad \frac{dD}{dt} = -n_0 + \frac{G}{B} + \frac{1}{G} \operatorname{cosec}\rho (1 - \cos\rho) \frac{\partial W}{\partial \rho}$$

where, under the simplifications made,

$$W = \langle U\rangle |_{l=0, \theta=\pi/2} = -\sigma\lambda_0 \{2(1 + \nu)(2 - \delta)\cos^2\rho + \delta f(\nu)(1 + \cos\rho)^2 \cos 2(D + \Psi_0)\} \tag{2.4}$$

$$\cos 2\Psi_0 = -\frac{1 - \nu}{2f(\nu)}, \quad \sin 2\Psi_0 = \frac{\sqrt{3}}{2} \frac{1 - \nu}{f(\nu)}, \quad f(\nu) = \sqrt{1 - \nu + \nu^2}$$

Equations (2.3) and (2.4) averaged according to the Delaunay scheme, have a complete system of first integrals

$$\frac{G^2}{2B} - G\cos\rho\, n_0 - W(\rho, D, \delta) = C_1, \quad G(1 - \cos\rho) = C_2, \quad h - h_0 = -n_0(t - t_0) + \int_{t_0}^{t} \frac{1}{G} \operatorname{cosec}\rho \frac{\partial W}{\partial \rho} dt \tag{2.5}$$

$$g - g_0 = \int_{t_0}^{t} \left\{\frac{G}{B} - \frac{1}{G} \operatorname{ctg}\rho \frac{\partial W}{\partial \rho}\right\} dt, \quad t - t_0 = \int_{D_0}^{D} \left\{-n_0 + \frac{G}{B} + \frac{1}{G} \operatorname{cosec}\rho (1 - \cos\rho) \frac{\partial W}{\partial \rho}\right\}^{-1} dD$$

Formulas (2.5) represent the general integral of the intermediate problem and contain a complete set of arbitrary constants C_1, C_2, h_0 and g_0. For the practical application of the intermediate rotational motion obtained it is also important that a general solution of the problem is constructed, i.e. that the elements G, ρ, g, h, D are represented as explicit functions of time.

3. Analysis of the resonant motions of a triaxial satellite. We introduce the resonant values G^* and ρ^* of the variables G and ρ by means of the formulas $G^* = Bn_0$, $\cos\rho^* = 1 - C_2/G^*$, and assume $G = G^* + \Delta G$. Then, using the first integrals defined by the first two formulas of (2.5), we obtain the following approximate expression for G (correct to $\sqrt{\sigma}$):

$$G = G^* [1 + k\sqrt{\sigma\Lambda}\,(1 + \cos\rho^*) \operatorname{cn} u], \quad \Lambda = \frac{3}{4} \frac{f(\nu)}{1 + \nu} \delta \tag{3.1}$$

Next, using the second formula of (2.5) we obtain the relationship $\rho(D)$, and this enables us to find the solution of the equation describing the Delaunay anomaly. Omitting for brevity the derivation, we give the solution of the averaged equations in terms of approximate formulas (retaining the basic terms of the order $\sim\sqrt{\sigma}$):

$$\cos\rho = \cos\rho^* + k\sqrt{\sigma\Lambda}\sin^2\rho^*\,\mathrm{cn}\,u\,, \quad h - h_0 = -n_0(t - t_0) + \sigma H(t - t_0) - \sqrt{\sigma\Lambda}\,\mathrm{zn}\,u \tag{3.2}$$

$$\sin\beta = \mathrm{dn}\,u,\ \cos\beta = -k\,\mathrm{sn}\,u,\ \beta = D + \Psi_0\,,\ u = \sqrt{\sigma\Lambda}(1 + \cos\rho^*)\,n_0(t - t_0),\ H = H_0 + H_1(1 - E/K)$$

$$H_0 = \frac{3}{8}\frac{1}{(1+\nu)}n_0[2(1+\nu)(2\delta - 1)\cos\rho^* + \delta f(\nu)(1 + \cos\rho^*)],\quad H_1 = -\Lambda n_0(1 + \cos\rho^*),\ g = \beta - h$$

$$0 < k^2 = \{(1 + \cos\rho^*)^2\Lambda - \lambda_0[2(1+\nu)(2 - \delta)\cos^2\rho^* + \delta f(\nu)(1 + \cos\rho^*)^2 - C]\}[(1 + \cos\rho^*)^2\Lambda]^{-1} < 1$$

Here k is the modulus of the elliptic functions in terms of which the motion in question is described, C is the reduced energy constant, K and E are complete elliptic integrals of the first and second kind and $\mathrm{sn}\,u, \mathrm{cn}\,u, \mathrm{dn}\,u, \mathrm{zn}\,u$ are the elliptic Jacobi functions.

Solution (3.2) describes a three-dimensional libration of a rigid body relative to the center of mass. Such a type of motion takes place when the initial conditions satisfy the inequality

$$|\beta_0^{\cdot}| < (1 + \cos\rho^*)\sqrt{\Lambda}\,|\sin\beta_0| \tag{3.3}$$

The resonant motion determined by the formulas (3.2) has a distinctive feature consisting of the fact that the vector G of kinetic moment coincides with the axis of inertia $S\eta$ during the whole motion. At the same time the vector G executes a slow secular motion with angular velocity of $\sim\sigma$. The secular motion is overlaid with resonant oscillations of amplitude of the order of $\sim\sqrt{\sigma}$. The trajectory of the vector G on the unit sphere describes the figure 8 /2/.

The rotational motion of the spacecraft has a long term periodicity. The period of its resonant oscillations is defined by the formula

$$T = T_0\frac{2K(k^2)}{\sqrt{\sigma\Lambda}(1 + \cos\rho^*)} \tag{3.4}$$

where $T_0 = 2\pi/n_0$ is the period of rotation of the basic bodies, and the period of a precessional motion of the vector G is $T = 2\pi/(\sigma H)$. We note that the solution of the averaged equations can be constructed in the form of series in powers of $\sqrt{\sigma}$ to any prescribed accuracy.

Using the present formulation of the problem we can investigate completely only two types of resonant motions, namely those with the commensurabilities $2n_0 = n_1^{(0)}$ and $n_0 = n_1^{(0)}$. Other types of commensurability can be studied within the framework of the restricted, elliptic three-body problem.

It should be noted that the resonant motions investigated include the motions generating periodic rotations of the spacecraft (rigid body). The latter solutions are characterized by specified initial values of the Andoyer variables /3/, namely:

$$l_0 = g_0 = 0,\ \ \pi/2,\ \pi,\ 3\pi/2,\ \ h_0 = h_{00} + k\pi/2 \quad (k = 0, 1, 2, 3)$$

$$\cos 2h_{00} = -\frac{1+\nu}{2\sqrt{1 - \nu + \nu^2}}\,,\quad \sin 2h_{00} = \frac{\sqrt{3}}{2}\frac{1-\nu}{\sqrt{1 - \nu + \nu^2}}\,,\quad \theta_0 = \pi/2,\quad \cos\rho_0 = \frac{\varepsilon_2\nu_2(1 - 2\delta - \varepsilon_1)}{2(1+\nu)(1 - 2\delta + 3\varepsilon_1) - \varepsilon_2\nu_2(1 - 2\delta - \varepsilon_1)}$$

$$\varepsilon_1 = \cos 2l_0 = \pm 1,\quad \varepsilon_2 = \cos 2g_0 = \pm 1,\quad \varepsilon_3 = 1\ (k = 0, 2),\quad \varepsilon_3 = -1\ (k = 1, 3),\quad \nu_2 = \frac{\sqrt{1 - \nu + \nu^2}}{1+\nu}\varepsilon_3$$

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